On Global Analysis of Duality Maps

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Abstract. A global analysis of duality transformations is presented. It is shown that duality between quantum field theories exists only when the geometrical structure of the quantum configuration spaces of the theories comply with certain precise conditions. Applications to S-dual actions and to T duality of string theories and D-branes are briefly discussed. It is shown that a new topological term in the dual open string actions is required. We also study an extension of the procedure to construct duality maps among abelian gauge theories to the non abelian case.

Duality transformations were introduced by Dirac and extended later on by Montonen and Olive. More recently were used by Seiberg and Witten [1] to relate the weak and strong coupling regime in the analysis of the low energy effective action of the N=2 SUSY SU(2) Yang-Mills. This approach to non-perturbative QFT was then introduced in string theory with spectacular success. It has been shown that the strong coupling regime of one string theory can be mapped to the weak coupling regime of another perturbatively different string theory, giving rise to a possible unification of all string theories in the context of a hypothetical M-theory. In this lecture, we analyse the duality transformations from a global point of view. This approach requires the introduction of a more general geometrical structure than the associated to line bundles over a general euclidean base manifold. We describe [2] the general structure of higher order line bundles, and define over them dual maps between theories described locally by p-forms.

Duality maps for theories with p-forms have been discussed in [3] and appear naturally in the description of D-brane theories. However, the global aspects of the configuration space of these local p-forms was never described. We show that the local analysis used in [3] is not enough to ensure quantum duality equivalence and give the necessary conditions to achieve it. The interesting result related to this global structure is that duality between theories of local p-forms and d-p-forms not only imply the quantization of couplings, the known generalized Dirac quantization condition, but also determine completely, from a global point of view,

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the configuration space of these local p-forms. These spaces are defined in terms of local p-forms with non trivial transitions on higher order bundles. In the first part of the lecture we explain the global approach for the Maxwell theory formulated over a general base manifold. We then give the general results concerning the higher order bundles and discuss some applications to D-brane theories. To do so we first consider the duality analysis for open bosonic strings. We prove that the dual open string action requires a new topological term in order to obtain the correct dual boundary conditions. In the second part of the lecture, we extend these ideas to the case of non abelian dualities.

I   ELECTROMAGNETIC DUALITY

The action of Maxwell theory over a 4-dim base manifold X, compact euclidean and orientable, is

$$I(F(A)) = \frac{1}{8\pi} \int_X d^4x \sqrt{g} [\frac{4\pi}{e^2} F^{mn} F_{mn} + i\frac{\theta}{4\pi}\frac{1}{2}\epsilon_{mnpq} F^{mn} F^{pq}] \tag{1}$$

where F is the curvature of a 1-form connection A on a U(1) bundle. This action may be rewritten in terms of the complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4i\pi}{e^2}$

$$I_\tau(A) = \frac{i}{8\pi} \int_X d^4x \sqrt{g} [\bar\tau F^+_{mn} F^{+mn} - \tau F^-_{mn} F^{-mn}] \ , \tag{2}$$

In order to construct the dual map we introduce an equivalent formulation to (1) or (2). We consider the action

$$I(\Omega) = \frac{i}{4\pi} [\bar\tau(\Omega^+, \Omega^+) - \tau(\Omega^-, \Omega^-)] \tag{3}$$

where Ω is a global 2-form satisfying the constraints

$$d\Omega = 0 \tag{4}$$

$$\oint_{\Sigma_2^I} \Omega = 2\pi n^I \tag{5}$$

The motivation to introduce the global constraint (5) is that by Weil's theorem constraints (4) and (5) ensure the existence of a unique complex line bundle and a connection on it -not necessarily unique- whose curvature is Ω. (3), (4) and (5) represent then an equivalent formulation to (1) in terms of the configuration space of 2-forms. It is relevant to determine how many connections giving Ω can be constructed for a given line bundle. In this case it is given by $H^1(X,R)/H^1(X,Z)$. The cohomology classes take into account all canonical gauge equivalent connections, while $H^1(X,Z)$ counts for the equivalence under "large" gauge transformations.

Having determined the exact correspondence between connections over line bundles and global 2-forms constrained by (4) and (5), we have to introduce now the

correct Lagrange multiplier. It must have also a precise global structure in order to account for the global constraint (5). It can be shown [2] that it may be expressed in terms of a 1-form connection V over the dual line bundle, provided summation over all dual bundles and all gauge inequivalent connections over every line bundle is performed in the functional integral. The resulting functional integral in terms of A must also be an integral on all line bundles and all gauge inequivalent connections on every line bundle. The important point to emphasize here is that the configuration spaces for the 1-form connections A and V are uniquely determined from the duality equivalence. In this sense the requirement to theory (1) of having a dual formulation determines completely the global structure of its configuration space. The quantization of magnetic charge is then only one consequence of this global structure. The resulting action after the introduction of the Lagrange multiplier is given by

$$\mathcal{I}(\Omega, V) = I(\Omega) + \frac{i}{2\pi} \int_X W(V) \wedge \Omega \tag{6}$$

.

From (6) one may integrate on V and regain (3), (4) and (5) and after solving the constraints (4) and (5) one obtains (1). We can also integrate on Ω and obtain the dual action in terms of V. The partition function of both quantum equivalent formulations is obtained in the standard way, with the known [3] result

$$\mathcal{Z}(\tau) = \mathcal{N} \tau^{-\frac{1}{2}B_2^-} \bar{\tau}^{-\frac{1}{2}B_2^+} \mathcal{Z}(-\frac{1}{\tau}) \tag{7}$$

where B_k^+ and B_k^- are the dimensions of the spaces of selfdual and antiselfdual k forms.

II DUALITY ON HIGHER ORDER BUNDLES

The generalization of the above construction may be analysed [2] by considering a globally defined p-form over X satisfying

$$dL_p = 0$$
$$\oint_{\Sigma_p^I} L_p = 2\pi n^I. \tag{8}$$

Let us consider $p = 3$. We take an open covering of $X : \{U_i, i \in I\}$. Without loosing generality we may consider every open set and its intersections to be contractible to a point . On U_i we have

$$L_3 = dB_i \tag{9}$$

and on $U_i \cap U_j \neq \phi$

$$dB_i = dB_j$$
$$B_i = B_j + d\eta_{ij} \tag{10}$$

where B_i is a 2-form with transitions given by (10), η_{ij} being a local 1-form defined on $U_i \cap U_j \neq \phi$. On $U_i \cap U_j \cap U_k$ we obtain

$$L_1 \equiv \eta_{ij} + \eta_{jk} + \eta_{ki}$$
$$dL_1 = 0 \tag{11}$$

From (8) we have

$$\int_{\Sigma_1} L_1 = 2\pi n, \tag{12}$$

where Σ_1 is a close curve on $U_i \cap U_j \cap U_k$. From (11) and (12) we obtain a 1-form L_1 defined over $U_i \cap U_j \cap U_k$ satisfying

$$dL_1 = 0 \tag{13}$$
$$\int_{\Sigma_1} L_1 = 2\pi n \tag{14}$$

which yields an uniform map from

$$U_i \cap U_j \cap U_k \to U(1) \tag{15}$$

The interesting property not present in the previous discussion is that the 1-cochain is now defined as

$$g : (i,j) \to g_{ij}(P,\mathcal{C}) \equiv \exp i \int_{\mathcal{C}} \eta_{ij} \tag{16}$$

where $\mathcal{C}$ is an open curve with end points O (a reference point) and P.

g associates to (i,j) a map $g_{ij}(P,\mathcal{C})$ from the path space over $U_i \cap U_j$ to the structure group U(1).

Notice that the 1-form η_{ij} cannot be integrated out to obtain a transition function as in the case of a line bundle. However, we have

$$\delta g_{ijk} = g_{ij} g_{jk} g_{ki} = \exp i \int_O^P L_1 \tag{17}$$

which is precisely the uniform map M previously defined in (15). (16) explicitly shows that the geometrical structure we are dealing with is not that of an usual $U(1)$bundle since the cocycle condition on the intersection of three open sets of the covering is not satisfied. Starting from transitions functions g_{ij} defined on the space of paths over $U_i \cap U_j$, and acting with the coboundary operator δ we obtain the 2-cochain (17) which is properly defined in the sense of Čech. We may go further and consider in the intersection of four open sets the action of the coboundary

operator δ on 2-cochains.We obtain on $U_i \cap U_j \cap U_k \cap U_l$ a 3-cocycle condition in the sense of $\check{C}ech$:

$$\delta g_{ijkl} = g_{ijk} g_{ijl}^{-1} g_{ikl} g_{jkl}^{-1} = \mathbb{1} \tag{18}$$

The construction leads then to local p-forms with non-trivial transitions defined by the higher order bundle. Having extended the geometrical structure of line bundles we may then formulate over them duality maps generalizing the electromagnetic duality.

The action for the local p-form A_p defined over open sets of a covering of X and with transitions defined over a higher order bundle is

$$S(A_p) = \frac{1}{2} \int_X F_{p+1} \wedge *F_{p+1} \tag{19}$$

where F_{p+1} is the curvature of A_p and the coupling constant has been reabsorbed in A_p.

Let us now consider its dual formulation. We introduce now the globally defined (p+1)-form L_{p+1} satisfying

$$dL_{p+1} = 0 \tag{20}$$

$$\oint_{\Sigma_p^I} L_{p+1} = \frac{2\pi n^I}{g_p}. \tag{21}$$

with action

$$S = \frac{1}{2} \int_X L_{p+1} \wedge *L_{p+1} \tag{22}$$

(20) and (21) ensure the existence of a bundle of order p+1 and a local p-form on it whose curvature is L_{p+1}.

The off-shell Lagrange problem of the above constrained system may be given by the action

$$S(L_{p+1}, V_{d-p-2}) = S(L_{p+1}) + i \int_X L_{p+1} \wedge W_{d-p-1}(V) \tag{23}$$

where V_{d-p-2} is a local (d-p-2)-form with transitions over a higher order bundle satisfying a (d-p-1)-cocycle condition and with coupling g_{d-p-2}.

Funtional integration on L_{p+1} yields

$$*L_{p+1} = -iW_{d-p-1} \tag{24}$$

where W_{d-p-1} is the curvature of V_{d-p-2}, and the dual action

$$S(V_{d-p-2}) = \frac{1}{2} \int_X W_{d-p-1}(V) \wedge *W_{d-p-1} \tag{25}$$

The quantum equivalence between the two dual actions follows once we integrate on all bundles of order p generalizing the electromagnetic duality previously shown. The quantization of charges arises directly from the global constraints needed for having a globally well defined higher order bundle. The configuration space of the local p-forms A_p and its dual are globally determined, they are defined over higher order bundles with cocycle condition of order $p+1$ which are classified by the integer numbers n^I associated to a basis of integer homology Σ_I on X. For a given bundle of order p+1 the different local antisymmetric fields up to gauge transformations are given by $H^p(X, U(1))/H^p(X, Z)$.

These local antisymmetric fields with non trivial transitions appear naturally in the description of D-branes. For example it has been [4] conjectured that the d=11 5-brane action is given by

$$S = -\frac{1}{2} \int_X d^6\xi \sqrt{-\gamma} [\gamma^{ij}\partial_i x^M \partial_j x^N \eta_{MN} + \frac{1}{2}\gamma^{il}\gamma^{jm}\gamma^{kn} F_{ijk} F_{lmn} - 4] \qquad (26)$$

where $F = dA$ is the self dual 3 form field strength of a local 2-form potential A which has to be defined over a bundle of order 3 if non trivial topological effects are expected. It would be interesting to determine completely from a geometrical point of view the moduli space of the self dual potentials over this higher order bundle. This problem is under study.

It is interesting to notice that dealing with D-brane theories, there are two different duality transformations involved. One is obtained by following the approach we have described previously with respect to the local 2-form A in (26). Because of the self duality condition, the curvature F_3 may be identified with W_3. The other duality arises by following the same approach but with a different interpretation for the global constraint, it is now related to the compactification condition on some of the coordinates on the target space. To show it in some detail we explain the duality transformation on the worldsheet of the string theory, and finally comment on the D=11 supermembrane, D=10 IIA Dirichlet supermembrane duality transformation which involve a compactification of χ^{11}, say, on S^1 and nontrivial line bundles over the worldvolume from the other side. That is, both kind of global constraints appears in the duality map.

III   T DUALITY

We discuss now the duality maps between first quantized string theories emphasizing the global constraint in the construction.

The string action is

$$S(\chi) = \frac{1}{2\alpha'} \int_\Sigma d^2\xi \sqrt{g} g^{ij} \partial_i \chi^\mu \partial_j \chi_\mu \qquad (27)$$

where g^{ij} is the world sheet metric and ξ^i , i=1,2 are the local coordinates of the Riemann surface Σ of a fixed topology. We analyse first the closed string theory

with one coordinate χ compactified over S^1. Associated to that coordinate we introduce a contrained 1-form $L = L_i d\xi^i$ satisfying

$$dL = 0 \tag{28}$$

$$\oint_{\mathcal{C}^I} L = 2\pi n^I R \tag{29}$$

where $\mathcal{C}^I$ denotes a basis of the integer homology of dimension 1 over the worldsheet. Contraint (28) implies L is a closed 1-form, while (29) ensures the compactification over S^1, R is the compactification radius. The solution to (28) and (29) is the string map $\chi(\xi^1, \xi^2)$. We introduce Lagrange multipliers associated to constraints (28) and (29) and obtain the quantum equivalent action

$$S(L, V) = \frac{1}{2\alpha'} \int_\Sigma L \wedge^* L + \frac{i}{\alpha'} \int_\Sigma L \wedge W(V) \tag{30}$$

where $V(\xi^1, \xi^2)$ is the dual map to $\chi(\xi^1, \xi^2)$:

$$W(V) = dV \tag{31}$$

$$\oint_{\mathcal{C}^J} = 2\pi m^J R' \tag{32}$$

(32) is uniquely determined to obtain quantum equivalence between $S(L, V)$ and $S(\chi)$.

Following the same arguments as in the S duality approach we obtain, in order to recover (29) from (30), after summation on all n in the functional integral,

$$R' = \frac{\alpha'}{R} \tag{33}$$

That is the dual radius arises directly from the off-shell construction of the dual action. From (30) we obtain the standard on-shell duality relation

$$^*L + iW = 0 \tag{34}$$

From (30) after functional integration on L we obtain

$$S(V) = \frac{1}{2\alpha'} \int_\Sigma W(V) \wedge^* W(V). \tag{35}$$

The duality between $L = d\chi$ and $W = dV$ is resumed in the global constraints (29), (32) and (34). Notice that (32) is uniquely determined from the off-shell construction while (29) implies the compactification of χ on S^1 of radius R. The quantum equivalence between (27) and (35) has been shown for any compact Riemann surface Σ hence the T-duality is valid order by order in the perturbative expansion of closed string theories. We now discuss the duality of open string theories. The standard open string boundary condition arises from (27) by considering the stationary points of $S(\chi)$. Its variation yields a boundary term

$$\left(\delta\chi\partial_i\chi n^i\right)\Big|_{\partial\Sigma} \tag{36}$$

It can be annihilated by assuming

$$\partial_i\chi^\mu n^i = 0 \tag{37}$$

This boundary condition together with the usual string field equation gives an stationary point of (27) with respect to the space of variations $\delta\chi$ which are arbitrary even on the boundary. If instead we consider the space of maps χ restricted by a boundary condition and look for a stationary point of (27) restricted to that space, then

$$\chi^\mu|_{\partial\Sigma} = cte \tag{38}$$

would be also a solution, since then $\delta\chi = 0$. In this case one can have even a mixture of Dirichlet and Newmann conditions on the boundary as an acceptable solution. We will discuss the construction of the dual string action on the first case and show that a topological action term has to be added to (35) in order to have a dual action whose stationary points yields the dual boundary condition to (37). Notice that from the duality relation (34) one obtains

$$n \cdot L = 0 \rightarrow t \cdot W = 0 \tag{39}$$

where t is tangent to the boundary. However from (35) if we consider arbitrary variations on the boundary we get

$$n \cdot W = 0 \tag{40}$$

We thus must modify (35) and consequently (30). We consider

$$\tilde{S}(L, V, Y) = S(L, V) + \frac{i}{\alpha'}\int_\Sigma F(Y) \wedge W(V) \tag{41}$$

where $F = dY$ and Y is a map onto S^1. The new term in the action is a pure topological one. It does not modify the field equations, only contributes to the boundary terms. All the local dependence of $Y(\sigma)$ can be gauged away, only the boundary contribution remains. The boundary terms in the variation of (41) are

$$(\delta V(L + F)|_{\partial\Sigma} = 0 \tag{42}$$
$$(\delta Y W)|_{\partial\Sigma} = 0 \tag{43}$$

which imply V=cte over any connected part of the boundary, and

$$(L + F(Y)) \cdot t|_{\partial\Sigma} \tag{44}$$

(44) does not add any restriction to L. It only determines $F(Y)$ on $\partial\Sigma$. After integration on L we obtain

$$\tilde{S}(V,Y) = \frac{1}{2\alpha'} \int_\Sigma W(V) \wedge^* W(V) + \frac{i}{\alpha'} \int_\Sigma F(Y) \wedge W(V) \tag{45}$$

We will consider now that V and χ are maps onto S^1 with compactification radius R' and R respectively. This implies that V=C on the $\sigma = 0$ boundary and $V = C + 2\pi n R'$ in the $\sigma = \pi$ boundary. We will show quantum equivalence between (27) with boundary condition $d\chi \cdot n = 0$ and (45) with boundary condition $dV \cdot t = 0$. Starting from (41) integration on V yields (27) and we are left with the boundary terms

$$\frac{i}{\alpha'} C \int_{\partial\Sigma} [L + F(Y)] + \frac{i}{\alpha'} 2\pi n R' \int_{\sigma=\pi} [L + F(Y)] \tag{46}$$

integration on C and summation on n yield

$$\delta \left(\int_{\partial\Sigma} [L + F(Y)] \right) \sum_m \delta \left(\frac{R'}{\alpha'} \int_{\sigma=\pi} [L + F(Y)] + 2\pi m \right) \tag{47}$$

They imply that

$$\int_{\sigma=\pi} L = [-Y(t_f) + Y(t_i)]_{\sigma=\pi} - 2\pi m R \tag{48}$$

which is the condition that χ is a map from the world sheet to S^1 with radius R. The construction yields

$$R' = \frac{\alpha'}{R}, \tag{49}$$

The global restriction is implemented here through the boundary conditions. We have shown that the dual action to the open string theory requires an additional topological term in the action in order to obtain the correct boundary condition.

The construction of global duality maps required then the implementation of a global constraint which in the case of S-duality ensures the existence of local p-forms with nontrivial transitions on a higher order bundle. In the case of T duality the global constraint is related to the compactification of one or several of the target coordinates. In the duality equivalence of the d=11 supermembrane and the d=10 IIA Dirichlet supermembrane the global constraint for the d=11 supermembrane is the compactification condition while the global constraint for the Dirichlet supermembrane ensures that the local 1-form A is a connection on a nontrivial line bundle over the world-volume. In the construction of duality maps between p-forms and (d-p-2)-forms the difficult but crucial step in the construction is the converse theorem that ensures that given a globally defined (p+1)-form L_{p+1} there exists a bundle of an order p and a local antisymmetric field with non-trivial transitions whose curvature is L_{p+1}. In the case of p=2 there is a very elegant construction of the higher order bundle in terms of Dixmier-Douady sheaves of groupoids [?].

The main result of the global analysis we have considered is that the existence of a quantum equivalent dual theory completely determines the configuration space of the potentials A_p and of its local dual forms V_{d-p-2}. The global constraint we have introduced are just the correct ones to describe the global structure of the configuration spaces. The geometrical description of these spaces allow an explicit formulation of the D-brane theories in terms of the potentials A_p, a necessary step for the quantization of the these theories.

IV   NON ABELIAN DUALITY

Duality maps between abelian gauge theories given by $U(1)$ connections on line bundles over a manifold X can be shown to exist by using a quantum equivalent formulation of the original theory in terms of closed 2-forms. This is expressed as a functional on the space of abelian 2-forms which must be constrained by non-local restrictions, namely, the requeriment of being closed and with integral periods, ensuring the existence of a 1-1 correspondence between the space of constrained 2-forms and the line bundles over X [3].This procedure has been successfully applied even to more general U(1) bundles [2] based on an extension of Weil's theorem to complex p-forms [5].Once the equivalence between the formulation in terms of the configuration space of abelian connections and that of the space of closed 2-forms is achieved, the latter is used to construct at the quantum level the dual gauge theory, by introducing dual Hodge-$\star$ forms through Lagrange multipliers proving the existence of non trivial relations between the partition functions of the abelian theory and its dual.

The purpose of this talk is to inquire on the possibility of extending the above procedure to the non abelian case. In the first place, we will begin by asking what conditions should be imposed on matrix-valued 2-forms over a manifold X so that we could produce something similar to Weil's theorem for non-abelian 2-forms, so that we could achieve an equivalence between the formulation of the theory on the configuration space of connections and the formulation on the space of 2-forms. This actually is a formidable problem still not solved but only to the level of conjectures [6]. In any case, we could try to see where failures lie.

The Bianchi identity for a matrix-valued 2-form Ω

$$\mathcal{D}\,\Omega = 0$$

is the first condition that comes to mind when looking for restrictions to implement, since curvatures for connections on fiber bundles satisfy it. But this, in general, does not assure even that Ω may be expressed locally in terms of any 1-form connection A as

$$\Omega = dA + A \wedge A$$

something equivalent to a Poincaré's lemma for "covariantly closed" forms does not hold. Moreover, even when we could express Ω in terms of A as above, on open

sets U_i of a covering of the manifold X, compatibility of the curvature-like 2-forms $\Omega(A_i)$ and $\Omega(A_j)$ on the intersection of two open sets U_i and U_j should imply that A_i and A_j are related by a well defined gauge transformation on the intersection of open sets. Simple calculations show that this is not the case. A_i and A_j could be related by some other more general transformations that no doubt include the mentioned gauge transformations i.e.

$$\Omega(A_i) = g^{-1}\Omega(A_j)\, g \stackrel{not}{\Longrightarrow} A_i = g^{-1}A_j\, g + g^{-1}dg$$

Obviously, we need more restrictive conditions to arrive to the necessary compatibility glueing for constructing globally well defined non abelian vector bundles.

It is well known [7], that a formulation of non abelian gauge theories has a rather simple expression on the space of loops as a trivial flat gauge theory. The main ingredient in this formulation is the use of the holonomy associated to each class of non abelian Lie algebra valued connections on a vector bundle. The use of holonomies is quite adequate since its non local character as a geometrical object carries a lot more information about the bundle than curvatures or connections. So, we should go to the loop space formulation and see whether it is possible to write some conditions that could characterize the non abelian bundles and look for a procedure to build the duality maps. In what follows, we suceed in proving half the task, for a more detailed discussion see [8].

For our purpose, instead of using the space of closed curves [7,9], we will consider a space of open curves $\mathcal{C}$ with fixed endpoints O, P over a compact manifold X. This will allow the construction of smoothly behaving mathematical objects like functionals, variations of functionals, 1-form connection functionals and so on, on open neighborhoods of the space of curves. Particularly, we avoid regularization problems in the definition of the gauge "potential" on path space.

First, any functional over this space will be denoted $\widetilde{\Phi}(\mathcal{C}_{\mathrm{O,P}})$ and a variation or increment of this functional due to a deformation on the curve leaving the endpoints fixed is defined as

$$\widetilde{\Delta}\widetilde{\Phi}(\mathcal{C}_{\mathrm{O,P}}) \equiv \widetilde{\Phi}(\mathcal{C}_{\mathrm{O,P}} + \delta\mathcal{C}_{\mathrm{O,P}}) - \widetilde{\Phi}(\mathcal{C}_{\mathrm{O,P}})$$

Deformations on the curves are smooth vector fields on open neighborhoods of X where the curve $\mathcal{C}_{\mathrm{O,P}}$ lies, tending to zero on the endpoints of the curve. We could relax this definition allowing non zero deformations on one of the endpoints but then we would need to impose a non linear condition to get the compatibility requirement on the patching of the vector bundle [8]. Our version of holonomy is H_A, the path ordered exponential of a 1-form connection A over X integrated over the open curve $\mathcal{C}$ i.e

$$H_A(\mathrm{O}, \mathrm{P}, \mathcal{C}) \equiv exp: \int_{\mathrm{O}}^{\mathrm{P}} A :$$

it becomes the ordinary holonomy when O and P are identified. $\widetilde{\mathcal{A}}(\mathcal{C}_{\mathrm{O,P}})$ denotes the 1-form connection functional acting on deformations S. It is obtained from H_A

$$\widetilde{\mathcal{A}}(\mathcal{C}_{\mathrm{O,P}}) = -\widetilde{\Delta}H_A \cdot H_A^{-1}$$

and may be expressed in terms of $F_{p(t')}$, the ordinary pointwise defined curvature 2-form associated to the connection A, as

$$\tilde{\mathcal{A}}(\mathcal{C}_{\mathrm{O,P}})[S] = \int_{\mathrm{O}}^{\mathrm{P}} H_A(\mathrm{O}, p(t'), \mathcal{C}) F_{p(t')}[T, S] H_A(\mathrm{O}, p(t'), \mathcal{C}^{-1}) dt'$$

where T is a vector field tangent to the curve $\mathcal{C}$, t' is a parameter along the curve and $p(t')$ is an ordinary point on the curve.

$\tilde{\mathcal{A}}(\mathcal{C})$ is defined for classes of equivalence of ordinary connections under gauge transformations, i.e it is gauge invariant up to elements of the structure group on the endpoints of the curve. This is a rather nice feature of working in path spaces.

We could continue and define also the curvature functional $\widetilde{\mathcal{F}}(\mathcal{C}, A)$ for the connection functional $\tilde{\mathcal{A}}$ in the usual manner

$$\widetilde{\mathcal{F}}(\mathcal{C}, A) = \tilde{\Delta}\tilde{\mathcal{A}}(\mathcal{C}) + \tilde{\mathcal{A}}(\mathcal{C}) \wedge \tilde{\mathcal{A}}(\mathcal{C})$$

for this free formulation of non abelian gauge theories, calculations show that

$$\widetilde{\mathcal{F}}(\mathcal{C}, A) = 0$$

and it is a gauge invariant statement.

The "covariant" derivative $\tilde{\mathcal{D}}$ may be also introduced as

$$\tilde{\mathcal{D}} \cdot \equiv \tilde{\Delta} \cdot + \tilde{\mathcal{A}} \wedge \cdot$$

In the case of the space of curves with two fixed endpoints, we need only to require compatibility of $\tilde{\mathcal{A}}(\mathcal{C}_{\mathrm{O,P}})$ on the intersecting neigborhoods

$$\tilde{\mathcal{A}}(\mathcal{C}_{\mathrm{O,P}})_i = \tilde{\mathcal{A}}(\mathcal{C}_{\mathrm{O,P}})_j$$

and get, in the same manner, that this is only possible if and only if

$$A_i = g^{-1}A_j \, g + g^{-1}d \, g.$$

So we have suceeded in the first step towards the construction of dual non abelian fields suggesting that the natural space for building up the dual maps are loop spaces or open curve spaces. Now, it rests to find a global condition equivalent to that of integral periods of the curvature 2-form for abelian gauge theories, that actually labels the different line bundles, i.e an equivalent Dirac quantization condition. We know that for particular $SU(2)$ bundles, there may be a splitting into the direct sum of two line bundles, for those bundles the usual Dirac quantization may suffice. In the space of paths then the restriction to be imposed would be that the ordinary curvature 2-form appearing in $\tilde{\mathcal{A}}(\mathcal{C}_{\mathrm{O,P}})$ would belong to the set of "diagonalizable" 2-form curvatures through a condition involving the intersection form and the non abelian topological charge associated to the second Chern class. This suggests that perhaps the global condition needed for non abelian gauge theories, at least for the case of $SU(N)$, involves the "quantization" of the topological

charge associated to the second Chern class. Once we find the exact condition to be imposed on the path space, the dual map for constructing dual gauge theories should be no problem since the operation that generalizes the Hodge-$\star$ operation for path spaces has already been defined at least on shell in [10] and improved in [8] in the sense that no regularization is needed. The partition function is also easily implemented in our formulation. A characterization of matrix-valued 2-forms for being curvatures of non abelian bundles has also recently been conjectured using partial differential equations on a loop space [6].

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